Solving the Twin's Paradox with Special Relativity

Guy Leckenby

Abstract

In this paper, we examine one of the most well known of special relativity's apparent paradoxes: the twin's paradox. It is regularly claimed that because this paradox contains accelerating frames that it does not belong in the domain of special relativity but must be solved via general relativity. We will demonstrate that this is not the case by considering to solutions to the paradox which only involve flat Minkowski spacetime. The first is the Doppler shift analysis which considers what each observer actually sees according to the relativistic Doppler equation which finds that both observers agree the travelling twin is younger. Then we solve it analytically by considering Rindler coordinates which quantify accelerating frames in special relativity. This will give us a numeric result which shows that the Earthbound twin ages the required amount during the turnaround according to the accelerated frame. Thus, by considering the appropriate framework, we demonstrate there is no paradox involved at all.

I. Introduction

The twin's paradox is probably the most well known of special relativity's 'paradoxes' and has continued to confuse fledgeling physicists since it was first conceived by Einstein, despite the fact it was solved almost as soon as it was constructed. The large majority of analyses, however, simply point out the most obvious logical flaw and move on without taking any time to examine the subtleties of the problem. In fact, the twin's paradox demonstrates many of the strange properties that come with a relativistic outlook. In this paper, we will examine the problem and solve it in several different ways. In doing so, I hope that the reader will be able to see how the varied pieces of special relativity come together cohesively to explain the observed phenomena.

II. The Twin's Paradox

Here's the problem. Consider two twins, Adelle and Chris. Adelle is adventurous and decides to jump on board the interstellar spaceship *HMS Joe is Great* where she will fly to a new colony around Tau Ceti, a star that is 12 light years away. The journey will be made at 60% of the speed of light which corresponds to a gamma factor of $\gamma(0.6c) = 1.25$. Adelle waves goodbye, jets off to Tau Ceti, turns around and comes home at the same speed reuniting with her brother on Earth. On her return, there's considerable confusion however. Chris has seen Adelle travel at $0.6c$

relative to Earth the entire time and thus undergoing time dilation by the gamma factor of 1.25; he thinks the 40-year journey has taken her 32 years. But because velocity is relative, Adelle has seen Chris moving at $0.6c$ so during her 32-year journey (length contraction has made the trip shorter for her, she sees his clock moving at 80% the rate of hers so Chris must be the younger one by 6.4 years! Now both twins think the other is younger — is this a paradox?

Well fortunately enough, no. This analysis has assumed symmetry in the problem when there is clear asymmetry. To return to Earth, Adelle has to undergo considerable acceleration in the Tau Ceti system which means she no longer occupies an inertial frame of reference. Chris felt no such acceleration. Therein lies the asymmetry that nullifies the paradox because time dilation is applied incorrectly to a non-inertial reference frame.

Many explanations of the paradox will simply stop here and say something along the lines of 'special relativity can only handle inertial reference frames. General relativity is required to deal with this accelerating frame, so we won't even bother'. This is not true however — special relativity **can** deal with accelerating frames. The difference is that general relativity treats all frames on equal footing whilst special relativity treats inertial and non-inertial frames differently. The only sense that special relativity is an approximation is that the generation of gravitational waves from an accelerating body are ignored.[1] However, in these problems, there are many more significant effects of acceleration that are considered negligible so this isn't really a problem.

The correct answer to the original question of who is younger is that Adelle is younger by the associated gamma factor. We will consider two solutions to the problem to explain this ageing difference, each coming from a different perspective. In the Doppler effect analysis, we will consider exactly what each observer sees and show how the relativistic Doppler effect accounts for the asymmetry and correctly predicts both ages when Adelle returns to Earth. Secondly, we will consider the hyperbolic motion of a uniformly accelerated frame and show that this predicts that Chris actually ages by the required factor during the acceleration of Adelle's turnaround.

III. The Doppler Effect Analysis

In this analysis, we will consider exactly what each observer sees throughout the journey. Suppose we equip both Adelle and Chris with very powerful telescopes and flashing clocks such that in each clock's proper time, it emits a flash of light each second. The relativistic Doppler effect discussed by Rindler[2]

1 Philip Gibbs, 'Can special relativity Handle Acceleration?' (1996, The Original Usenet Physics FAQ)<http://math.ucr.edu/home/baez/physics/Relativity/SR/acceleration.html>.

2 Wolfgang Rindler, *Relativity: Special, General, and Cosmological* (Oxford University Press, 2006) 78–81.

claims that

$$\frac{\nu_0}{\nu_S} = \sqrt{\frac{1-\beta}{1+\beta}} \tag{1}$$

where ν_0 is the frequency observed, ν_S is the frequency of the source and $\beta = v/c$ is the velocity as a fraction of the speed of light assuming recession is positive. In the case we are considering, we've set $\nu_S = 1$ Hz in our flashing clocks. For this analysis, we'll consider the case in which the acceleration is instantaneous and/or negligible. Although this is unrealistic, it is a limiting case of the theory and hence should produce correct predictions.[3]

So let's consider what Chris observes. As Adelle travels away from him, the frequency of her flashing clock on the outbound leg, as given by the relativistic Doppler equation, is $\nu_O = 0.5$ Hz. He knows the outbound journey takes 16 years of Adelle's time at the speed she is going, but because of the information rate, Chris won't see Adelle reach Tau Ceti until his clock reads $\tau_O = 32$ years. However, on the return journey when Adelle's velocity is reversed, Chris will observe her flashing clock with a frequency of $\nu_I = 2$. The time he experiences until Adelle returns is then $\tau_I = 8$ years. Hence when Adelle returns to Earth in 32 years, Chris will have aged 40 years predicting that she aged exactly 80% the time he has.

Now let's consider what Adelle observes. Chris travels away from her so again, the frequency observed is $\nu_O = 0.5$ Hz. The moment of turnaround for Adelle is 16 years into her 32-year journey. Hence she will also observe Chris to have aged $\tau_O = 8$ years at the time of turnaround. However, after the turnaround she observes a frequency of $\nu_I = 2$ for 16 years, so the time that elapses for Chris on her return journey is $\tau_I = 32$ years. Thus Adelle correctly observes that Chris has aged 40 years whilst she has only aged 32.

So by considering exactly what each observer sees via the relativistic Doppler effect, the paradox has evaporated and both observers agree that Adelle is younger.[4] Furthermore, we have achieved this without ever considering acceleration. There is symmetry on both the outbound and inbound journeys; both twins observe the same red- and blue-shift factors. The fundamental asymmetry that allowed the correct conclusion is that the transition from red- to blue-shift occurred at Adelle's turnaround. She shares equal observing time for both the red- and blue-shifts. However, Chris doesn't see Adelle turn around until she's almost home and thus the increased frequency from the blue-shift can't catch up, giving the desired result.[5] This result is particularly well demonstrated by Figure 1 where the signal that Adelle observes is seen in Figure 1a whilst the signal Chris receives is seen in Figure 1b.

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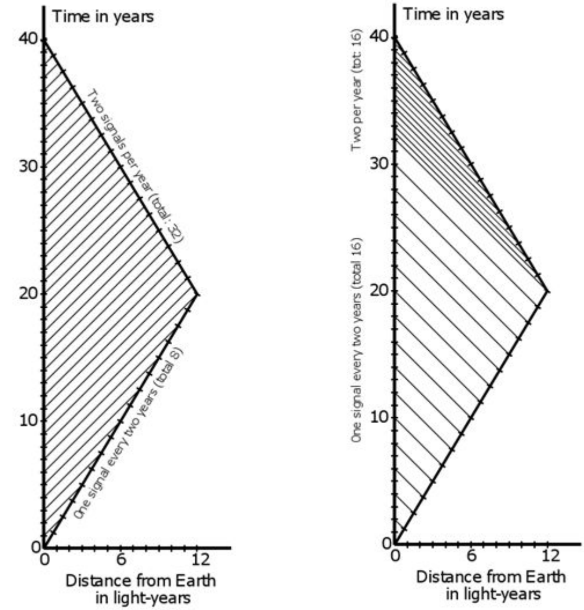
3 Michael Weiss, 'The Twin Paradox.' (The Original Usenet Physics FAQ) <http://math.ucr.edu/home/baez/physics/Relativity/SR/TwinParadox/twin_paradox.html>.

4 Paul Davies, *About Time* (Simon & Schuster, 1995) 59–65.

5 Weiss, above n 3.



(a) Adelle receives the signal from Chris' flashing clock.

(b) Chris receives the signal from Adelle's flashing clock.

Figure 1: With Adelle travelling at $0.6c$, the spacetime diagram shows the flashes of light received by both Adelle and Chris throughout the journey. As evident, Adelle sees Chris age 40 years whilst Chris sees Adelle age 32 years.[6]

The sceptic will say at this point 'well hang on, regardless of what they observe, couldn't the twins *deduce* that the other's clock is running slower than theirs? How can you have a difference in what they theoretically deduce and what they actually observe?'[7] This is the Time Gap Objection and it says that if Adelle calculates rather than observes, she will deduce that Chris magically ages 6 years during the instantaneous turnaround.

This isn't really a problem because that deduction is a result of her changing inertial reference frames. The inbound reference frame says the turnaround happens at $t = 12.8$ years for Chris whilst the outbound reference frame says the turnaround happens at $t = 27.2$ years. The apparent time gap of 14.4 years is simply an accounting error induced by changing inertial reference frames.[8] We will show

6 SlidePlayer, 'Slide show Notes_06.ppt' (2009) <http://slideplayer.com/slide/5056347/#>.

7 Davies, above n 4.

8 Weiss, above n 3.

in the final section of the paper that this accounting error is predicted exactly by the acceleration Adelle undergoes.

Now at this point, you might be thinking 'I like how the twins' observations match up and that the deductions about the time dilation fit together coherently, but it still seems like some magic happens during the acceleration.' Let us deal with that acceleration head on.

IV. Hyperbolic Motion

Special relativity can deal with any problem occurring in flat Minkowski spacetime. In this section we will consider uniform acceleration as it is mathematically much simpler. However, any type of acceleration can be dealt with by the same framework.

Given a position 4-vector $\mathbf{r}$, we define the 4-velocity to be $\mathbf{u} = \mathrm{d}\mathbf{r}/\mathrm{d}\tau$ and correspondingly the 4-acceleration to be $\mathbf{a} = \mathrm{d}\mathbf{u}/\mathrm{d}\tau$ where τ is the proper time. Consider two results which we have proved in Assignment 8; $\mathbf{u}^2 = c^2$ and $\mathbf{a}\cdot\mathbf{u} = 0$. Now consider some observer who initially begins in an inertial frame and then feels a constant acceleration g in the x_1 direction and for the remaining spacelike components, assume $x_2 = x_3 = 0$. Then the equations of motion for the observer naturally are

$$\begin{aligned} \frac{\mathrm{d}t}{\mathrm{d}\tau} &= u_0, & \frac{\mathrm{d}x_1}{\mathrm{d}\tau} &= u_1; \\ \frac{\mathrm{d}u_0}{\mathrm{d}\tau} &= a_0, & \frac{\mathrm{d}u_1}{\mathrm{d}\tau} &= a_1. \end{aligned} \tag{2}$$

Furthermore from the results of Assignment 8, we have

$$\begin{aligned} u_\mu u^\mu &= c^2 \\ u_\mu a^\mu &= u_0 a_0 - u_1 a_1 = 0 \\ a_\mu a^\mu &= g^2. \end{aligned} \tag{3}$$

Solving these algebraic equations for the acceleration, we produce two linear differential equations

$$a_0 = \frac{\mathrm{d}u_0}{\mathrm{d}\tau} = \frac{g u_1}{c}, \qquad a_1 = \frac{\mathrm{d}u_1}{\mathrm{d}\tau} = \frac{g u_0}{c} \tag{4}$$

which can be solved immediately[9]. Choosing an appropriate origin, the solutions are

$$ct = \frac{c^2}{g}\sinh\left(\frac{g\tau}{c}\right), \qquad x_1 = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right). \tag{5}$$

These equations of motion are termed *hyperbolic motion* because the resulting worldline is the hyperbola $x_1^2 - (ct)^2 = (c^2/g)^2$ in a spacetime diagram,[10] as

9 Charles Misner, Kip Thorne, and John Wheeler, *Gravitation* (W.H. Freeman and Company, 1973) 165–73.

10 Ibid.

demonstrated in Figure 2. Note that only the positive half of the parabola is shown because that is what the parametric plot considers. If the integration constants and origin had been chosen otherwise, the negative half could have been the one selected. Thus we have the position 4-vector of a uniformly accelerating observer in terms of their initial rest frame.

Despite the fact that this derivation was very simple, it demonstrates all of the crucial principles to explaining why the acceleration causes Chris to age. Consider two timelike events A and B which occur at times t_A and t_B in the inertial reference such that $t_B - t_A = \Delta t$. Then the path between the two events undergoing a uniform acceleration g occupies a hyperbolic path and the proper time for that observer is given by $\Delta\tau = c/g \sinh^{-1}(g\Delta t/c)$ from Equation (5). Producing a plot of the proper time as a function of g yields Figure 3. As is clearly evident, any acceleration will cause a decrease in the proper time that gets larger with increasing acceleration. That is, of all the timelike paths between events A and B, the one with the longest lapse in proper time is the unaccelerated one.[11]

This demonstrates that if an observer undergoes acceleration, their proper time will be smaller than when compared to the stationary observer. Chris feels no acceleration so he can correctly deduce Adelle's age on her return just from time dilation. However, Adelle underwent acceleration during her turnaround, so she has to account for the fact that Chris's clock will run faster than hers during the acceleration. Let's quantitatively prove this.

V. Rindler Coordinates

Whilst hyperbolic motion neatly solved the conceptual elements of the twin's paradox by showing us that the unaccelerated frame always has the longest proper time lapse between two events, it does not calculate a numerical solution to the originally posed problem. The parametrisations given in Equation (5) have assumed convenient choices of origin and initial conditions to yield the simple result. This will not work for our case because Adelle undergoes her acceleration far away from the origin at very specific initial conditions. Hence we have to generalise our concept of hyperbolic motion with *Rindler coordinates*.[12]

Consider a reference frame with coordinates (t, x) undergoing uniform acceleration and an associated inertial frame with coordinates (T, X) (note that (t, x) is simply shorthand for the full 4-vector (t, x, y, z) because $y = Y$ and $z = Z$ so are not of interest). Rindler coordinates are the coordinates (x, t) and are interesting because they are associated with the accelerating frame; for more on Rindler coordinates, refer to Appendix A. The transformations between these two coordinate systems

[11] Ibid.

[12] Benjamin Knorr, 'Uniform Relativistic Acceleration' (2010) <http://www.physik.uni-leipzig.de/ schiller/ed10/Uniform relativistic acceleration.pdf>.

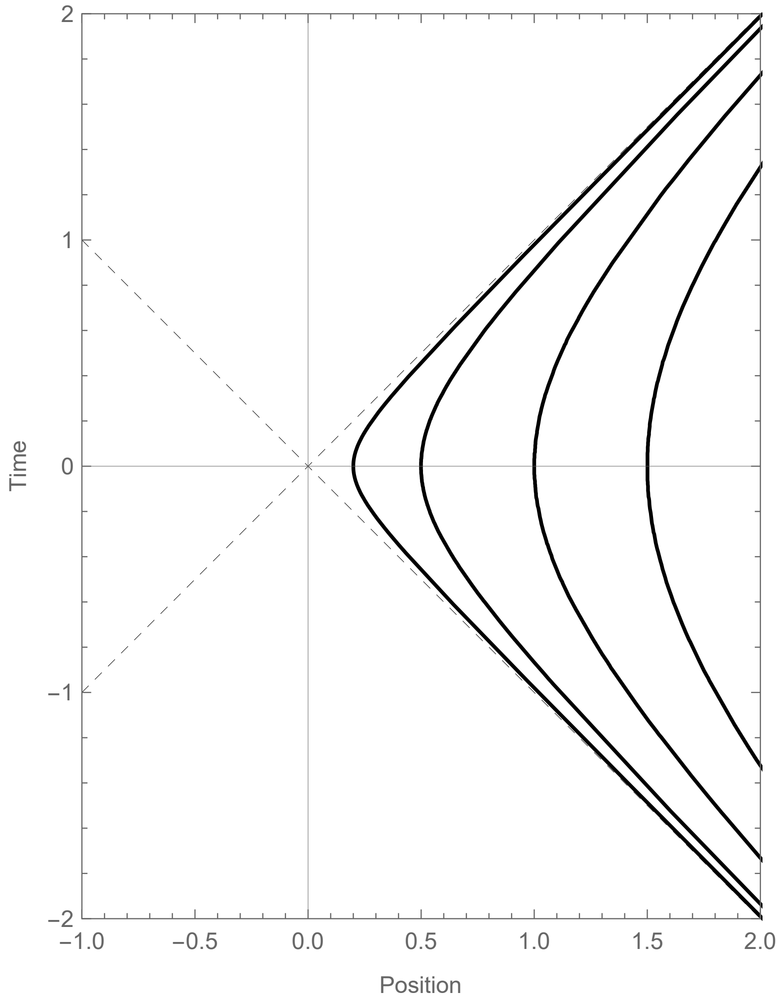


Figure 2: The worldline of a uniformly accelerating observer given by $x_1^2 - (ct)^2 = (c^2/g)^2$. The steepness of the curve depends on the value of acceleration g. Smaller accelerations are to the right.

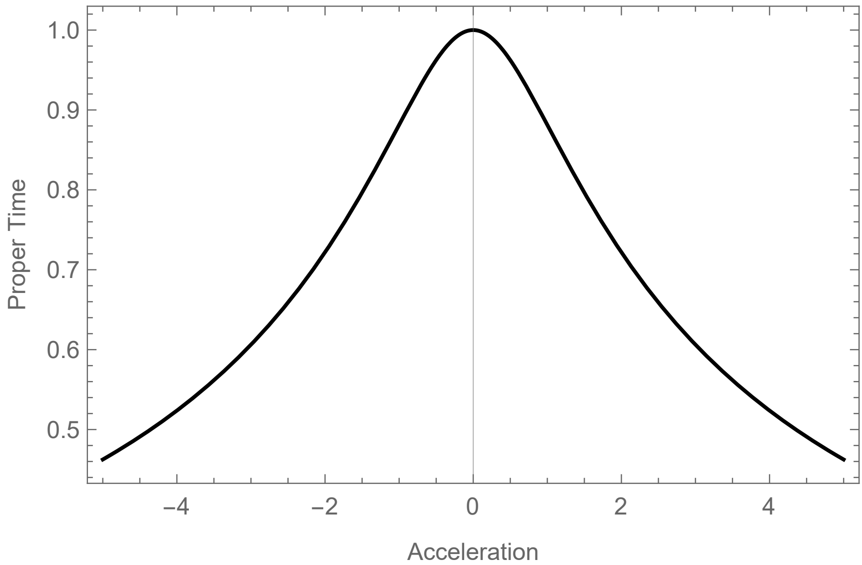


Figure 3: The proper time $\Delta\tau$ of a uniformly accelerated observer as a function of acceleration g assuming $\Delta t = c = 1$.

are

$$
\begin{aligned}
X &= \left(x + \frac{c^2}{g}\right)\sinh\left(\frac{g(t-t_0)}{c}\right) + X_0 - \frac{c^2}{g} \\
cT &= \left(x + \frac{c^2}{g}\right)\cosh\left(\frac{g(t-t_0)}{c}\right) + cT_0
\end{aligned}
\tag{6}
$$

and the associated inverse transformations are

$$
\begin{aligned}
x &= \sqrt{\left(X - X_0 + \frac{c^2}{g}\right)^2 - c^2(T-T_0)^2} - \frac{c^2}{a} \\
ct &= \frac{c^2}{g}\tanh^{-1}\left(\frac{c(T-T_0)}{X - X_0 + \frac{c^2}{g}}\right) + ct_0
\end{aligned}
\tag{7}
$$

where x is the spatial offset of the origin of the accelerated frame with respect to the inertial frame, t_0 is the time offset, and X_0 is the spatial offset of the moving particle.[13] Equation (6) is a generalisation of Equation (5) and accounts for not only spatial separations but also temporal separations of the accelerated frames.

13 Ibid.

According to Grøn,[14] Iorio[15] and Knorr,[16] Equation (7) gives rise to a line element of

$$ds^2 = \left(1 + \frac{gX}{c^2}\right)^2 c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{8}$$

The general physical interpretation of a line element in a timelike interval is that it is the proper time of an observer following the worldline of two nearby events separated by (dt, dx, dy, dz).[17] That is $ds^2 = c^2 d\tau^2$. Thus we conclude that for an inertial observer sitting at a distance $X = h$ from the accelerating observer, the observed proper time of the observer is

$$\Delta\tau = \left(1 + \frac{gh}{c^2}\right) \Delta t. \tag{9}$$

VI. Solving the Paradox

Thus we have all the tools required to analytically solve the twin's paradox with special relativistic acceleration. We have analysed hyperbolic motion to show that acceleration causes a decrease in the proper time of the accelerated observer and invoked Rindler coordinates to fully quantify that for an observer of the accelerated motion.

Now let's consider the original problem. To simplify the analysis, we will assume that $g \to \infty$ and because this is a limiting case, the analysis should still hold up.[18] Let's consider what Chris deduces. He sees Adelle travel at $0.6c$ to Tau Ceti 12 light years away, instantaneously turns around and comes back. The journey takes 40 years of his time, but due to time dilation, Adelle only ages 32 years.

Now let's consider what Adelle sees. During the 9.6 ly flight which takes her 16 years, she sees Chris age only 12.8 years due to time dilation (similarly on the return journey). Her turnaround, in terms of the inertial frame, will take her $\Delta t = 2v/g$. Because she views herself at rest, she sees Chris as the one accelerating and predicts he will age by

$$\Delta\tau = \left(1 + \frac{gh}{c}\right) \frac{2v}{g} = \frac{2v}{g} + \frac{2hv}{c^2}. \tag{10}$$

In the limit that $g \to \infty$, this simplifies to $\Delta\tau = 2hv/c^2$. Since we have $h = 12\,\text{ly}$ and $v = 0.6c$, we find that Chris ages $\Delta\tau = 14.4$ years during the turnaround. Combining this with the 25.6 time dilated years she sees during the journey, she

14 Øyvind Grøn, 'The twin paradox in the theory of relativity' (2006) 27 *European Journal of Physics* 885–9.

15 Lorenzo Iorio, 'An analytical treatment of the Clock Paradox in the framework of the Special and General Theories of Relativity' (2004, ArXiv.org) <https://arxiv.org/abs/physics/0405038>.

16 Knorr, above n 12.

17 Grøn, above n 14.

18 Weiss, above n 3.

predicts Chris will be 40 years old on this return, precisely consistent with what he experiences.

Thus we have quantitatively solved the paradox! For small acceleration windows such that we can assume h to be constant, the proper time experienced by the inertial observer can be calculated from the accelerated frame. This was only possible because of the line element deduced from the Rindler coordinates which in turn were deduced from the hyperbolic motion.

VII. Conclusion

In this paper, we have analysed the twin's paradox from several angles. In most physical cases, we considered the limiting case of infinite acceleration and instantaneous turnaround to simplify the arithmetic. First, we considered the Doppler shift analysis which showed that if we consider exactly what each observer sees, then there is no paradox at all and both observers agree that Adelle is younger by the appropriate time dilated factor, 8 years in our example. This is reassuring because the observers cannot see an instantaneous turnaround so if they did not agree, the following physical arguments would not work.

Adelle and Chris are smart physicists, however, and we showed that as they are aware of the Doppler shift, they can calculate around it and still deduce that there should be a paradox. That's when we appealed to uniform acceleration to deal with the asymmetry in the problem. We found that an accelerated observer will always experience less proper time than an inertial observer and hence as Adelle undergoes the acceleration of the turnaround, she will see Chris age faster than her. However, because the analysis was simple, we couldn't quantify this ageing. Hence we referred to Rindler coordinates and the associated line element. Whilst a direct proof is beyond the scope of this paper, they allowed us to deduce the proper time experienced by the inertial observer whilst the accelerated observer undergoes the acceleration. This allowed us to show that if Adelle appropriately accounts for her acceleration, she will see Chris age exactly the required amount to deduce that he will be 8 years older on her return.

Thus we have numerically solved the twin's paradox and shown it is no paradox at all. Furthermore, we have achieved this without the need to refer to general relativity and demonstrated that because this is a Minkowski spacetime problem, special relativity can handle it.

A. Appendix: Rindler coordinates

Rindler coordinates arise as one possible set of coordinates to describe an accelerating frame of reference in Special Relativity. The transformations from an inertial reference frame described by (T, X, Y, Z) to a uniformly accelerating frame described by (t, x, y, z) are given by Equations (6) and (7). The Minkowski line element is given by Equation (8).

This transformation is only valid inside the *Rindler wedge* which is the space given by $0 < X < \infty$, $-X < T < X$.[19] The reason for this is because a Rindler observer experiences an event horizon along the worldline $T = \pm X$,[20] any point outside of this quadrant is inaccessible to the accelerating and hence they cannot see past it. The Rindler wedge for $g = 1$ is seen in Figure 4.

Figure 4 shows hyperbolic lines passing through different x values. For observers on these lines to maintain equal distance between themselves and others on the line, they must experience uniform acceleration of varying magnitudes.[21] That is for an observer who is spatially ahead of the observer who occupies $x = 1$, to maintain equal distance in Rindler coordinates he must experience a lesser acceleration. Similarly, those who are spatially behind must have greater acceleration; the observer occupying $x = 0$ has $g = \infty$. Any observer could define the Rindler coordinate as within the coordinate system, all possible acceleration values occupy a worldline. By convention, $g = 1$ is chosen for simplicity.[22]

Rindler coordinates are difficult in several regimes. Firstly they give rise to different physical observations, in particular the Unruh effect which claims that the accelerated observer will view radiation emanating from the Rindler horizon in a similar manner to Hawking radiation.[23] Furthermore, Rindler observers have no shared concept of simultaneity,[24] something that is expected of an inertial coordinate system. This is a result of the fact that the laws of physics are different in an accelerated frame in Special Relativity which makes them very troublesome to deal with.

19 Wikipedia, 'Rindler Coordinates' (2016) <https://en.wikipedia.org/wiki/Rindler_coordinates>.

20 Viktor Toth, 'What is the significance of Rindler coordinates?' (2016, Quora) <https://www.quora.com/What-is-the-significance-of-Rindler-coordinates>.

21 Austin Gleeson, 'Chapter 6: Uniform Acceleration' <https://web2.ph.utexas.edu/~gleeson/RelativityNotesChapter6.pdf>.

22 Wikipedia, above n 19.

23 Toth, above n 20.

24 Gleeson, above n 22.

25 'Rindler coordinate chart (family of parametric curves) in TikZ' (2013, TEX) <http://tex.stackexchange.com/questions/105822/rindler-coordinate-chart-family-of-parametric-curves-in-tikz>.

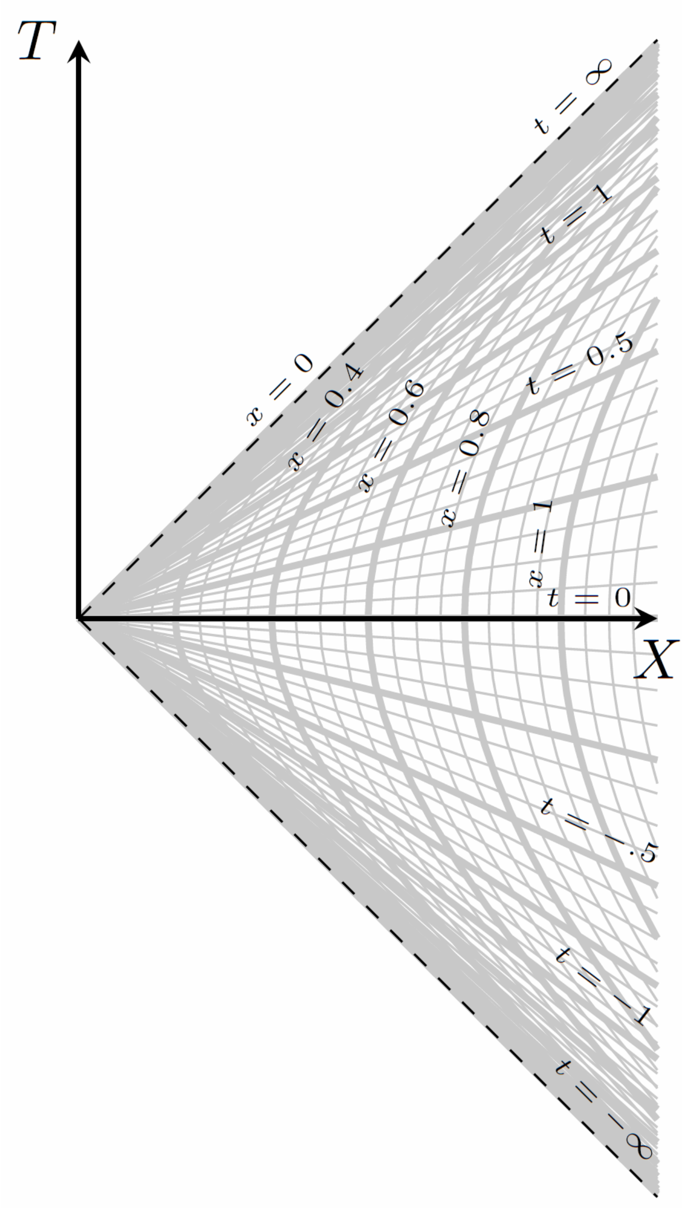


Figure 4: The Rindler coordinate chart otherwise known as the Rindler wedge. It demonstrates lines of simultaneity in Rindler coordinates.[25]

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